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LETTER TO THE EDITOR

Motion of four-dimensional rigid body around a fixed point: an elementary approach I

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Abstract
The goal of this letter is to give an elementary approach to the solution of Euler–Frahm equations for the Manakov four-dimensional case. For this, we use the Kötter approach and some results from the original papers by Schottky, Weber and Caspary. We hope that such an approach will be useful for the solution of the problem of an n-dimensional top.

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The equations of motion for a rigid body in a four-dimensional Euclidean space with a fixed point coinciding with the centre of mass (and also for the n-dimensional case) are the generalization of famous Euler's equations. They were found first by Frahm [1][2] and they have the form

$$\dot{l}_{ij} = \sum_{k=1}^{4}(l_{ik}\omega_{kj} - \omega_{ik}l_{kj}), \qquad \omega_{ij} = c_{ij}l_{ij}, \quad l_{ij} = -l_{ji}, \quad i, j = 1, \ldots, 4. \tag{1}$$

Here $c_{ij} = I_{ij}^{-1}$, the dot denotes the derivative with respect to time t, and l_{ik}, ω_{jk} and I_{ik} are components of angular momentum, angular velocity and principal momenta of inertia tensors, respectively.

In this letter we consider a completely integrable Manakov's case [3], when quantities c_{ij} have the form[3]

$$c_{ij} = \frac{b_i - b_j}{a_i - a_j}. \tag{2}$$

In a number of papers (see references in [4–6], and in the recent book [7]) the so-called method of linearization on the Jacobian of a spectral curve defined by the characteristic polynomial of one of the matrix in the Lax pair was used. However, as was mentioned in [6], (see also [5])

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[2] The problem of generalization of Euler's equations was posed by Cayley [2].
[3] Note that for the 'physical' rigid body $c_{ij} = I_{ij}^{-1}$, $I_{ij} = I_i + I_j$. In this letter we consider a general integrable case when quantities c_{ij} and I_{ij} are arbitrary.

'this approach has remained unsatisfactory; indeed (i) finding such families of Lax pairs often requires just as much ingenuity and luck as to actually solve the problem; (ii) it often conceals the actual geometry of the problem'.

Except these papers it should be mentioned in papers by Dubrovin [8, 9] where the solution is given in terms of theta functions of three variables. However, as is well known, the final formulae should contain the transcendental from more simple functional class, namely theta functions of two variables and such solution on best author's knowledge was absent.

So, in this letter we return to the original Schottky–Kötter approach [10, 11]. In our opinion, this elementary and natural approach is more adequate for the problem under consideration. We hope that it will also be useful for the more complicated problem of an n-dimensional top at $n > 4$.

Recall that in paper [10] the problem under consideration was reduced to the Clebsch problem [12] of the motion of a rigid body in an ideal fluid[4]. For the special cases, the last problem was integrated explicitly by Weber [14] and by Kötter [11].

However, the Clebsch problem is related not to the $so(4)$ Lie algebra but to the $e(3)$ Lie algebra—the Lie algebra of motion of the three-dimensional Euclidean space. Hence, it is important to extend the Schottky–Kötter approach to give the solution in the $so(4)$ covariant form. Here we give such an approach[5].

Note that equations (1) are Hamiltonian with respect to the Poisson structure for the $so(4)$ Lie algebra—the Lie algebra of rotations of the four-dimensional Euclidean space,

$$\{l_{ij}, l_{km}\} = l_{im}\delta_{jk} - l_{ik}\delta_{jm} + l_{jk}\delta_{im} - l_{jm}\delta_{ik}. \tag{3}$$

The Hamiltonian is given by the formula

$$H = \frac{1}{2}\sum_{j<k}^{4} c_{jk} l_{jk}^2, \tag{4}$$

where quantities c_{ij} are given by formula (2), and equations (1) may be written in the form

$$\dot{l}_{jk} = \{H, l_{jk}\}. \tag{5}$$

Let us recall that equations (1) have four integrals of motion

$$H_0 = l_{12}l_{34} + l_{23}l_{14} + l_{31}l_{24} = h_0, \tag{6}$$

$$H_1 = \sum_{j<k}^{4} l_{jk}^2 = h_1, \qquad H_2 = \sum_{j<k}^{4} a_{jk} l_{jk}^2 = h_2, \qquad H_3 = \sum_{j<k}^{4} b_{jk} l_{jk}^2 = h_3, \tag{7}$$

where

$$a_{jk} = (a_1 + a_2 + a_3 + a_4 - a_j - a_k), \qquad b_{jk} = (a_1a_2a_3a_4)/(a_ja_k).$$

Note that H_0 and H_1 are the Casimir functions of the $so(4)$-Poisson structure, and the manifold $\mathcal{M}_h$ defined by equations (6)–(7) is an affine part of two-dimensional Abelian manifold (see the appendix by Mumford to paper [4])[6]. Then formula (5) defines a Hamiltonian vector field on $\mathcal{M}_h$.

The main result of this note is the following one: by elementary means, it is shown that the dynamical variables $l_{jk}(t)$ are expressed in terms of Abelian functions $f_{j4}(u_1, u_2)$, $f_{kl}(u_1, u_2)$, $f_0(u_1, u_2)$ and $g(u_1, u_2)$ related to the genus two algebraic curve

$$y^2 = R(x) = \prod_{j=0}^{4}(x - d_j), \qquad d_0 = 0, \qquad d_4 = d_1d_2d_3, \tag{8}$$

with arguments depending linearly on time.

4 This result was rediscovered one century later in paper [13].

5 A special $so(4)$ case with tensor l_{jk} of rank 2 was integrated explicitly by Moser [15].

6 I am grateful to A N Tyurin for the explanation of algebraic geometry related to this appendix.

Theorem 1. *Solution of equations (1) has the form*

$$m_j = l_{kl} = g(u_1, u_2)(\alpha_j f_{kl}(u_1, u_2) + \beta_j f_{j4}(u_1, u_2)), \tag{9}$$

$$n_j = l_{j4} = g(u_1, u_2)(\gamma_j f_{kl}(u_1, u_2) + \delta_j f_{j4}(u_1, u_2)). \tag{10}$$

Here (j, k, l) *is a cyclic permutation of* $(1, 2, 3)$, $\alpha_j, \beta_j, \gamma_j, \delta_j$ *and* d_j *are algebraic functions of integrals of motion and quantities* a_j *and* b_k*. Explicit expressions for them are given by (24)–(26), (34), (35), (41) and (42).*

Proof. The key problem is the 'uniformization' of the manifold $\mathcal{M}_h$, i.e., the finding of the 'good' coordinates on it. The proof consists of several steps. □

Following Kötter [11] and using the linear change of variables m_j and n_j to new variables ξ_j and η_j, we transform equation (7) to the more appropriate form:

$$\sum_{j=1}^{3}\left(\xi_j^2 + \eta_j^2\right) = 0, \qquad \sum_{j=1}^{3}\xi_j\eta_j = 0, \qquad \sum_{j=1}^{3}\left(d_j\xi_j^2 + d_j^{-1}\eta_j^2\right) = 0. \tag{11}$$

For this, following Schottky [7], let us introduce the three-dimensional vector $\mathbf{l}(s)$ depending on the parameter s:

$$\mathbf{l}(s) = (l_1(s), l_2(s), l_3(s)), \qquad l_j(s) = \sqrt{s_{j4}}m_j + \sqrt{s_{kl}}n_j, \tag{12}$$

where

$$m_j = l_{kl}, \qquad n_j = l_{j4}, \qquad s_{jk} = s_{jk}(s) = (s - a_j)(s - a_k), \tag{13}$$

and $\{j, k, l\}$ is a cyclic permutation of $\{1, 2, 3\}$. It is easy to check that the function

$$f(s) = \mathbf{l}(s)^2 = \sum_{j=1}^{3} l_j(s)l_j(s) \tag{14}$$

does not depend on time. So, it is the generating function of integrals of motion

$$f(s) = h_1s^2 - h_2s + h_3 + 2h_0\sqrt{G(s)}, \qquad G(s) = \prod_{j=1}^{4}(s - a_j). \tag{15}$$

From formulae (12) and (14) it is easy to get the Lax representation[7]

$$\dot{L}(s) = [L(s), M(s)], \tag{16}$$

where $L(s)$ and $M(s)$ are antisymmetric matrices of third order corresponding to vectors $\mathbf{l}(s)$ and $\mathbf{m}(s)$,

$$\mathbf{m}(s) = (m_1(s), m_2(s), m_3(s)), \qquad m_j(s) = \sqrt{s_{kl}}m_j + \sqrt{s_{j4}}n_j, \tag{17}$$

$$L(s) = \begin{pmatrix} 0 & l_3 & -l_2 \\ -l_3 & 0 & l_1 \\ l_2 & -l_1 & 0 \end{pmatrix}, \qquad M(s) = \begin{pmatrix} 0 & m_3 & -m_2 \\ -m_3 & 0 & m_1 \\ m_2 & -m_1 & 0 \end{pmatrix}. \tag{18}$$

The equation $f(s) = 0$ is equivalent to the algebraic equation of fourth degree $F(s) = \prod_{j=1}^{4}(s - s_j) = 0$, where

$$F(s) = \left[(h_1s^2 - h_2s + h_3)^2 - 4h_0^2G(s)\right]\big/\left(H_1^2 - 4h_0^2\right). \tag{19}$$

[7] However, this representation is not needed for the proof of the theorem. For the generalization of such representation for the n-dimensional case, see [16].

This equation has four roots s_1, s_2, s_3 and s_4 that, in general, are complex ones. To them correspond four complex vectors

$$\mathbf{l}^{(p)} = \mathbf{l}(s_p)/\sqrt{F'(s_p)}, \qquad p = 1, 2, 3, 4, \tag{20}$$

(here $F'(s)$ is the derivative of $F(s)$) but only two of them, for example $\mathbf{l}^{(1)}$ and $\mathbf{l}^{(2)}$, are linearly independent, and

$$\begin{aligned} (\mathbf{l}^{(p)})^2 &= \sum_{k=1}^{3} \left(l_k^{(p)}\right)^2 = 0, \qquad p = 1, 2, 3, 4, \\ \sum_{p=1}^{4} \left(l_k^{(p)}\right)^2 &= 0, \qquad k = 1, 2, 3. \end{aligned} \tag{21}$$

Let us also introduce the vectors ξ and η by the formulae[8]

$$\xi_j = l_j^{(1)} + \mathrm{i} l_j^{(2)}, \qquad \eta_j = l_j^{(1)} - \mathrm{i} l_j^{(2)}. \tag{22}$$

Using (12) and (22) we may express m_j and n_j in terms of ξ_j and η_j

$$m_j = \alpha_j \xi_j + \beta_j \eta_j, \qquad n_j = \gamma_j \xi_j + \delta_j \eta_j, \tag{23}$$

where

$$\begin{aligned} \alpha_j &= \frac{\sqrt{s_{kl}^{(2)}/F'(s_2)} - \mathrm{i}\sqrt{s_{kl}^{(1)}/F'(s_1)}}{\Delta_j^{(3)}}, \qquad \beta_j = \frac{\sqrt{s_{kl}^{(2)}/F'(s_2)} + \mathrm{i}\sqrt{s_{kl}^{(1)}/F'(s_1)}}{\Delta_j^{(3)}}, \\ \gamma_j &= \frac{\sqrt{s_{j4}^{(2)}/F'(s_2)} - \mathrm{i}\sqrt{s_{j4}^{(1)}/F'(s_1)}}{\Delta_j^{(3)}}, \qquad \delta_j = \frac{\sqrt{s_{j4}^{(2)}/F'(s_2)} + \mathrm{i}\sqrt{s_{j4}^{(1)}/F'(s_1)}}{\Delta_j^{(3)}}, \end{aligned} \tag{24}$$

$$\Delta_j^{(p)} = \frac{\sqrt{s_{j4}^{(q)} s_{kl}^{(r)} - s_{j4}^{(r)} s_{kl}^{(q)}}}{\sqrt{F'(s_q) F'(s_r)}}, \qquad s_{kl}^{(p)} = s_{kl}(s_p). \tag{25}$$

Here (j, k, l) and (p, q, r) are cyclic permutations of (1, 2, 3).

Now it is easy to check that equations (7) take the form of three Kötter's quadrics (11), where

$$\sqrt{d_j} = \frac{\Delta_j^{(1)} - \mathrm{i}\Delta_j^{(2)}}{\Delta_j^{(3)}}, \qquad \frac{1}{\sqrt{d_j}} = -\frac{\Delta_j^{(1)} + \mathrm{i}\Delta_j^{(2)}}{\Delta_j^{(3)}}. \tag{26}$$

Following [11], let us show that the manifold defined by equations (11) may be 'uniformized' by means of the Weierstrass Wurzelfunctionen related to the hyperelliptic curve (8) that are defined as

$$P_j(z_1, z_2) = \sqrt{(z_1 - d_j)(z_2 - d_j)}, \qquad j, k = 0, 1, 2, 3, 4, \tag{27}$$

$$P_{jk}(z_1, z_2) = \frac{P_j P_k}{(z_1 - z_2)} \left(\frac{\sqrt{R(z_1)}}{(z_1 - d_j)(z_1 - d_k)} - \frac{\sqrt{R(z_2)}}{(z_2 - d_j)(z_2 - d_k)} \right). \tag{28}$$

[8] As was noted by Yu N Fedorov, there is a relation of these vectors to the problem of geodesics on two-dimensional ellipsoid with half-axes $\sqrt{d_j}$, $j = 1, 2, 3$. Namely, ξ may be considered as a tangent vector to geodesics and $\mathrm{i}\eta$ as a normal vector to this geodesics.

These 16 functions $P_j(z_1, z_2)$ and $P_{jk}(z_1, z_2)$ satisfy a lot of identities. All of them may be obtained from definitions (27) and (28) (for details, see [11, 14, 17])[9]. Here we give only a few of them which are useful to us:

$$\sum_{j=1}^{3} c_j \left(\frac{P_{kl}^2}{(s-d_k)(s-d_l)} + \frac{P_{j4}^2}{(s-d_j)(s-d_4)} \right) = \frac{s}{\prod_{j=1}^{4}(s-d_j)}, \tag{29}$$

$$\sum_{j=1}^{3} \tilde{c}_j P_{j4}^2 = d_4, \qquad \sum_{j=1}^{3} d_j \tilde{c}_j P_{kl}^2 = P_0^2, \tag{30}$$

$$\sum_{j=1}^{3} c_j P_{j4} P_{kl} = 0, \qquad \sum_{j=1}^{3} \tilde{c}_j P_{j4} P_{kl} = -P_0, \tag{31}$$

$$\sum_{j=1}^{3} c_j \left(d_j^{-1} P_{j4}^2 + d_j P_{kl}^2 \right) = 0, \tag{32}$$

where

$$\tilde{c}_j = \frac{1}{(d_j - d_k)(d_j - d_l)}, \qquad c_j = \frac{d_j - d_4}{(d_j - d_k)(d_j - d_l)}. \tag{33}$$

Note that with the algebraic curve (8) are related not only the functions $P_j(z_1, z_2)$ and $P_{jk}(z_1, z_2)$ but also the standard theta functions of two variables u_1 and u_2. It is well known that $P_j(z_1, z_2)$ and $P_{jk}(z_1, z_2)$ up to constant factors C_j and C_{jk} are the ratio of the theta functions with half-integer theta characteristics related to the hyperelliptic curve (8) (see for example [9, 14, 15] where the explicit expressions for C_j and C_{jk} can be found)[10]

$$P_j(z_1, z_2) = f_j(u_1, u_2) = C_j \frac{\theta_j(u_1, u_2)}{\theta(u_1, u_2)}, \qquad P_{kl}(z_1, z_2) = f_{kl}(u_1, u_2) = C_{kl} \frac{\theta_{kl}(u_1, u_2)}{\theta(u_1, u_2)}, \tag{34}$$

$$\begin{aligned}
\theta_{23}(u_1, u_2) &= \theta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u_1, u_2), & \theta_{31}(u_1, u_2) &= \theta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u_1, u_2), \\
\theta_{12}(u_1, u_2) &= \theta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u_1, u_2), & \theta_{14}(u_1, u_2) &= \theta \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (u_1, u_2), \\
\theta_{24}(u_1, u_2) &= \theta \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (u_1, u_2), & \theta_{34}(u_1, u_2) &= \theta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u_1, u_2), \\
\theta_0(u_1, u_2) &= \theta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (u_1, u_2), & \theta(u_1, u_2) &= \theta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u_1, u_2).
\end{aligned} \tag{35}$$

Here

$$\theta(u_1, u_2) = \sum_{n_1, n_2} \exp\{\mathrm{i}\pi(n_1(2u_1 + n_1\tau_{11} + n_2\tau_{12}) + n_2(2u_2 + n_1\tau_{21} + n_2\tau_{22}))\}, \tag{36}$$

and τ_{jk} are elements of period matrix of curve (8).

The comparison of (11) with (29)–(33) shows that

$$\xi_j = \sqrt{c_j}\, g P_{kl}, \qquad \eta_j = \sqrt{c_j}\, g P_{j4}, \tag{37}$$

where g is an unknown function.

[9] See also modern survey [18].

[10] Here we give just one series of such expressions. For the relative other series, see [11].

The rest of the proof is the uniformization of equation (6).

Let us substitute expressions (23) for m_j and n_j into equation (6). Then by using (24) and (25) we transform it to the form

$$H_0 = \sum_{j=1}^{3} \left(A_j\left(\xi_j^2 - \eta_j^2\right) + B_j\xi_j\eta_j\right) = h_0, \tag{38}$$

where

$$A_j = \alpha + \beta d_j + \gamma d_j{}^{-1}, \qquad B_j = \delta\left(d_j + d_j^{-1}\right). \tag{39}$$

Here α, β, γ and δ are algebraic functions of h_0, h_1, h_2, h_3, a_j and b_j.

This sum may be calculated by using (24), (29)–(33). The result is

$$H_0 = \frac{(1-\varepsilon P_0)^2}{4\varepsilon d_4} g^2 = h_0, \tag{40}$$

where

$$\varepsilon = \frac{\sqrt{d_4}\left(\sqrt{(s_3-s_1)(s_2-s_4)} - \sqrt{(s_2-s_3)(s_1-s_4)}\right)}{\sqrt{(s_1-s_2)(s_3-s_4)}}. \tag{41}$$

From this we obtain

$$g = c(1-\varepsilon f_0)^{-1}, \qquad \xi_j = g\sqrt{c_j} f_{kl}, \qquad \eta_j = g\sqrt{c_j} f_{j4}, \quad c = \text{const}. \tag{42}$$

The fact of linear dependence of arguments u_1 and u_2 on time t follows from the algebraic geometrical approach [7] as from the old Kötter approach [11].

This completes the proof of the theorem.

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References

[1] Frahm F 1874 Über gewisse Differentialgleichungen *Math. Ann.* **8** 35–44

[2] Cayley A 1846 Sur quelques propriétés des détérminants gauches *J. Reine Angew. Math.* **32** 119–23

[3] Manakov S V 1977 Note on the integration of Euler's equations of the dynamics of an n-dimensional rigid body *Funct. Anal. Appl.* **10** 328–9

[4] Adler M and van Moerbeke P 1982 The algebraic integrability of geodesic flow on $SO(4)$ *Inv. Math.* **67** 297–326 (with appendix by D Mumford, pp 326–31)

[5] Haine L 1983 Geodesic flow on $SO(4)$ and abelian surfaces *Math. Ann.* **263** 435–72

[6] Adler M and van Moerbeke P 1988 The Kowalewski and Hénon-Heiles motions as Manakov geodesic flows on $SO(4)$—a two-dimensional family of Lax pairs *Commun. Math. Phys.* **113** 659–700

[7] Adler M, van Moerbeke P and Vanhaecke P 2004 *Algebraic Integrability, Painlevé Geometry and Lie Algebras* (Berlin: Springer)

[8] Dubrovin B A 1977 Completely integrable Hamiltonian systems associated with matrix operators and abelian varieties *Funct. Anal. Appl.* **11** 28–41

[9] Dubrovin B A 1981 Theta functions and non-linear equations *Russ. Math. Surv.* **36** 11–92

[10] Schottky F 1891 Über das analytische problem der rotation eines starren Körpers im Raume von vier dimensionen *Sitzungber. König. Preuss. Akad. Wiss. zu Berlin* 227–32

[11] Kötter F 1892 Über die Bewegung eines festen Körpers in einer Flüssigkeit I. II. *J. Reine Angew. Math.* **109** 51–81, 89

[12] Clebsch A 1871 Über die Bewegung eines Körpers in einer Flüssigkeit *Math. Ann.* **3** 238–61

[13] Bobenko A I 1986 Euler equations on the Lie algebras $e(3)$ and $so(4)$. Isomorphism of integrable cases *Funct. Anal. Appl.* **20** 53–6

[14] Weber H 1878 Anwendung der Thetafunctionen zweiter Veränderlich-er auf die theorie der Bewegung eines festen Körpers in einer Flüssig-keit *Math. Ann.* **14** 173–206

[15] Moser Ju 1980 Geometry of quadrics and spectral theory *Chern Symposium 1979* pp 147–88

[16] Fedorov Yu N 2000 Integrable systems, Poisson pencils, and hyperelliptic Lax pairs *Reg. Chaot. Dyn.* **5** 171–80

[17] Caspary F 1893 Sur une nouvelle manière d'établir les relations algébriques qui ont lieu entre les fonctions hyperelliptiques de première espèce *Ann. Sci. Ec. Norm. Sup.* **10** 253–94

[18] Bukhstaber V M, Enolskii V Z and Leikin D V 1997 Hyperelliptic Kleinian functions and applications *Solitons, Geometry, and Topology: on Crossroad Am. Math. Soc. Transl. Ser. 2* vol 179 (Providence, RI: American Mathematical Society) pp 1–33